Numerical PDE Methods for Pricing the American Put Option

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An **option** is a type of derivative contract that gives the holder the right, but not the obligation, to buy or sell an underlying security at a certain price (**strike price**) and time (**expiry time**) in the future. Specifically a **put** option allows the holder to sell.

An **European** option only allows exercise of the right at the expiry date. This form of options admit an analytical solution formula.

An **American** option allows exercise at any time before and including the expiry date. This type of options does not admit a formula, therefore we resort to numerical methods.
Assume the asset price $S$ follows

$$dS = \mu Sdt + \sigma SdW$$

(1)

where $\mu$ is the drift, $\sigma$ is the volatility, and $W$ is the standard Wiener process.

For simplification of later equations, a change of variable $\tau = T - t$ is introduced. It can be shown that, if $S$ follows (1), the price $V(S, \tau)$ of an European option satisfies the Black-Scholes (BS) equation

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \equiv \mathcal{L}V$$

(2)

where $r$ is the risk-free interest rate.
The price of an option at expiry ($\tau = 0$) is determined by a payoff function $\Phi(S)$

$$V_{put}(S, \tau)\big|_{\tau=0} = \Phi_{put}(S) = \max(K - S, 0).$$

(3)

The $S$ variable, considered as space variable, spans the domain $[0, \infty)$. The prices of European put options at the left and right boundary of the spatial domain are given by

$$V_{put}(0, \tau) = Ke^{-r\tau}, \quad V_{put}(S, \tau) \xrightarrow{S\to\infty} 0$$

(4)

respectively.
For American options, there is an additional key constraint due to the ability to exercise at anytime before expiry $T$, which adds complexity to the problem:

$$V(S, \tau) \geq \Phi(S), \quad 0 \leq \tau \leq T. \quad (5)$$

This leads to a decision boundary at $S_f(\tau)$, where for each $\tau$ it is optimal to exercise on one side of the boundary, and optimal to hold on the other.
More specifically, the restriction can be reformulated as a linear complementarity problem (LCP)

\[
\frac{\partial V}{\partial \tau} - \mathcal{L}V \geq 0, \\
V - \Phi \geq 0, \\
(\frac{\partial V}{\partial \tau} - \mathcal{L}V)(V - \Phi) = 0.
\]

(6)

with new boundary conditions

\[
V_{put}(0, \tau) = K, \quad V_{put}(S, \tau) \xrightarrow{S \rightarrow \infty} 0.
\]

(7)
Figure: The price of an American put option follows different shapes on the two sides of the free boundary $S_f(\tau) = 51.57$. 
Instead of working with the full continuous price function $V(S, \tau)$, the function is sampled at specific points that form a grid,

$$V_{i,j} = V(S_i, \tau_j)$$

where a uniform grid is defined as

$$h_S = \frac{S_{\text{max}}}{N_S}, \quad S_i = ih_S, \quad i = 0, 1, \ldots, N_S$$

$$h_\tau = \frac{T}{N_\tau}, \quad \tau_j = jh_\tau, \quad j = 0, 1, \ldots, N_\tau.$$

The boundary condition to the right is approximated for large $S_{\text{max}}$

$$V_{\text{put}}(S_{\text{max}}, \tau_j) \approx 0.$$

We consider non-uniform grids later.
Finite Difference Method

**Figure:** A sample uniform discretization grid with $S_{\text{max}} = 800$, $T = 0.25$, $N_S = 5$, and $N_T = 5$. 
Finite Difference Method

Given the discretization grid points, the derivatives can be approximated by

\[
\frac{\partial V}{\partial \tau} \bigg|_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{h_{\tau}} + O(h_{\tau}) \quad (11a)
\]

\[
\frac{\partial V}{\partial \tau} \bigg|_{i,j+1} = \frac{V_{i,j+1} - V_{i,j}}{h_{\tau}} + O(h_{\tau}) \quad (11b)
\]

\[
\frac{\partial V}{\partial S} \bigg|_{i,j} = \frac{V_{i+1,j} - V_{i-1,j}}{2h_{S}} + O(h_{S}^2) \quad (11c)
\]

\[
\frac{\partial^2 V}{\partial S^2} \bigg|_{i,j} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h_{S}^2} + O(h_{S}^2). \quad (11d)
\]

In the above discretization schemes, \(O(h_{\tau})\) and \(O(h_{S}^2)\) represent the residual terms.
Similarly, the $\mathcal{L} V$ operator can be discretized as

$$
\mathcal{L} V|_{i,j} = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right)_{i,j} \\
= \frac{1}{2} \sigma^2 S_i \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{2h_S} + rS_i \frac{V_{i+1,j} - V_{i-1,j}}{2h_S} - rV_{i,j} + O(h_S^2)
$$

$$
= c_{i,i+1} V_{i+1,j} + c_{i,i} V_{i,j} + c_{i,i-1} V_{i-1,j} + O(h_S^2)
$$

(12)

where $c_{i,k}$ are scalars
Finite Difference Method

The previous equation can be vectorized, and rewritten in matrix form as

\[ \mathcal{L} \mathbf{V}_j = \mathbf{C} \mathbf{V}_j + \mathbf{D}_j + O(h_S^2) \] (13)

with

\[
\mathbf{V}_j = \begin{bmatrix} V_{1,j} & V_{2,j} & \cdots & V_{N_S-1,j} \end{bmatrix}^T
\]

\[
\mathbf{D}_j = \begin{bmatrix} c_{1,0} V_{0,j} & 0 & \cdots & 0 & c_{N_S-1,N_S} V_{N_S,j} \end{bmatrix}^T
\]

\[
\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & 0 & 0 & \cdots & 0 \\
 c_{2,1} & c_{2,2} & c_{2,3} & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & 0 & c_{N_S-1,N_S-2} & c_{N_S-1,N_S-1} \end{bmatrix}
\]

where the value of \( \mathbf{D}_j \) is given by the boundary conditions, therefore known a priori.
To summarize we converted the Black-Scholes PDE into a linear system of matrices

\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV
\]

\[
\Rightarrow C_{im} V_{j+1} + D_{im,j+1} = C_{ex} V_j + D_{ex,j} + h\tau O(h\tau + h_S^2)
\]  

(15)

where

\[
C_{im} = I - \frac{1}{2} h\tau C, \quad D_{im,j+1} = -\frac{1}{2} h\tau D_{j+1}
\]

\[
C_{ex} = I + \frac{1}{2} h\tau C, \quad D_{ex,j} = \frac{1}{2} h\tau D_j.
\]  

(16)
The discretization of (6) also yields a matrix form for the LCP as

\[
(C_{im}V_{j+1} + D_{im,j+1}) - (C_{ex}V_j + D_{ex,j}) \geq 0,
\]

\[
V_j - \Phi \geq 0,
\]

\[
[(C_{im}V_{j+1} + D_{im,j+1}) - (C_{ex}V_j + D_{ex,j})] \cdot [V_j - \Phi] = 0
\]

(17)
Choice of Grids

Figure: Typical solutions to the Black-Scholes PDE when strike price $K = 100$. 
**Choice of Grids**

**Figure:** A sample non-uniform price discretization grid with $S_{\text{max}} = 800$, $K = 100$, $N_S = 50$, and $a = 0.4$ under the $W_1(S)$ scheme combined with a uniform time discretization with $T = 0.25$ and $N_T = 10$. 
Choice of Grids

Non-Uniform Discretization Grid

**Figure:** A sample non-uniform price discretization grid with $S_{\text{max}} = 800$, $K = 100$, $N_S = 50$, and $a = 0.4$ under the $W_1(S)$ scheme combined with an adaptive non-uniform time discretization with $T = 0.25$, $N_T = 10$, and $d_{\text{norm}} = 0.2$.
The discretization of (6) also yields a matrix form for the LCP as

$$
(C_{im} V_{j+1} + D_{im,j+1}) - (C_{ex} V_j + D_{ex,j}) \geq 0, \quad V_j - \Phi \geq 0, \\
[(C_{im} V_{j+1} + D_{im,j+1}) - (C_{ex} V_j + D_{ex,j})] \cdot [V_j - \Phi] = 0
$$

(18)
Penalty Iteration vs. Operator Splitting

Penalty - First Guess
Solve matrix system

Operator Splitting
Solve matrix system

Continued Iteration
Solve matrix system

Finish

Approximate LCP
No matrix system

$V_j^k$
Discrete Penalty

The penalty method in Forsyth and Vetzal (2002) adds a term that punishes the difference arising from not satisfying the constraint. The formulation results in the relation

\[
\frac{\partial V}{\partial \tau} = \mathcal{L} V + \rho \max(\Phi - V, 0) \tag{19}
\]

where \( \rho \) is the penalty parameter, and is typically chosen to be a large positive number. In the discrete case, a penalty term is placed on the next step at \( V_{j+1} \), resulting in the relation

\[
[C_{im} + P_{j+1}] V_{j+1} + D_{im,j+1} = C_{ex} V_{j} + D_{ex,j} + P_{j+1} \Phi \tag{20}
\]

where \( P_{j+1} \) is the penalty matrix, with the \((i, n)\)th element defined as

\[
[P_{j}]_{i,n} = \begin{cases} 
\rho & \text{if } i = n \text{ and } V_{i,j} < \Phi \\
0 & \text{otherwise}
\end{cases} \tag{21}
\]
Discrete Penalty

Let the iteration occur at the \((j + 1)\)th time step, define \(V^k\) as the \(k\)th estimate of \(V_{j+1}\), and \(P^k\) as the penalty for \(V^k\).

The iteration algorithm then follows:

*Penalty Iteration for American Options*

Initialize \(V^0 = V_j\)

For \(k = 1, 2, \cdots\)

\[
\begin{align*}
\text{solve} \quad & \left[ C_{im} + P^k \right] V^k + D_{im,j+1} = C_{ex} V_j + D_{ex,j} + P^k \Phi \\
\text{if} \quad & \left( \max_i \frac{|V^{k+1} - V^k|}{\max(1, |V^{k+1}|)} < \frac{1}{\rho} \right) \quad \text{or} \quad [P^{k+1} = P^k] \quad \text{quit}
\end{align*}
\]

EndFor

\(V_{j+1} = V^k\)
Operator Splitting

In contrast to the penalty iteration, the operator splitting method used in Ikonen and Toivanen (2004) is a direct solver. Similar to the penalty method, operator splitting also adds a term to resolve the inequality arising from the LCP \( \frac{\partial V}{\partial \tau} - \mathcal{L} V \geq 0 \).

The auxiliary term is defined as

\[
\lambda \equiv \frac{\partial V}{\partial \tau} - \mathcal{L} V
\]  

(23)

Once again, the discretized LCP can be rewritten in matrix form

\[
(C_{im} V_{j+1} + D_{im,j+1}) - (C_{ex} V_j + D_{ex,j}) - h\tau \lambda_{j+1} = 0
\]  

(24)

where \( V_{j+1} \) and \( \lambda_{j+1} \) are both unknown vectors.
Operator Splitting

Instead of solving the previous equation, an intermediate term $U_j$ is introduced such that,

$$
(C_{im}U_{j+1} + D_{im,j+1}) - (C_{ex}V_j + D_{ex,j}) - h_{\tau}\lambda_j = 0 \quad (25a)
$$

$$
C_{im}(V_{j+1} - U_{j+1}) - h_{\tau}(\lambda_{j+1} - \lambda_j) = 0 \quad (25b)
$$

Next (25b) is approximated by assuming the unknowns in $V_{j+1}$ are independent, hence $C_{im}$ becomes the identity matrix, and we get

$$
V_{j+1} - U_{j+1} - h_{\tau}(\lambda_{j+1} - \lambda_j) = 0 \quad (26)
$$
As a result of the simplification, the LCP constraints force each $\lambda_{i,j+1}$ to take on one of two values

$$
\lambda_{i,j+1} = \begin{cases} 
0 & \text{if } V_{i,j+1} > \Phi(S_i) \\
\lambda_{i,j} - \frac{1}{h_\tau} [U_{i,j+1} - \Phi(S_i)] & \text{if } V_{i,j+1} = \Phi(S_i)
\end{cases}
$$

Hence, we can find the intermediate term $U_{j+1}$, the auxiliary term $\lambda_{i,j+1}$, and the price term $V_{j+1}$ explicitly

$$
U_{j+1} = C_{im}^{-1} \left[ C_{ex} V_j + D_{ex,j} + D_{im,j+1} + h_\tau \lambda_j \right]
$$

$$
\lambda_{i,j+1} = \max \left( \lambda_{i,j} - \frac{1}{h_\tau} [U_{i,j+1} - \Phi(S_i)], 0 \right)
$$

$$
V_{j+1} = U_{j+1} + h_\tau (\lambda_{j+1} - \lambda_j)
$$
Numerical Experiments were conducted with the following parameters on an American Put option:

\[
\begin{align*}
K &= 100 \\
T &= 0.25 \\
\sigma &= 0.8 \\
r &= 0.1 \\
S_{\text{max}} &= 800 \\
\rho &= 1e6
\end{align*}
\]
**Results - Choice of Methods**

**Figure:** Convergence Property of the Penalty Iteration and Operator Splitting methods with Uniform grids.
Results - Choice of Grids

Figure: Convergence Property of the Penalty Iteration with Uniform and Non-uniform grids.
Future Work

Higher Dimensions
Since the penalty iteration requires more than one solution of a linear system per time step, its computational cost is expected to scale faster in multiple dimensions than the operator splitting method, which requires only one linear system solution per time step.

Optimal Grid
There is no available guideline to choose the optimal parameters for a non-uniform grid.

Other Derivative Products
Asian options, barrier options etc.
Questions or Comments?

Operator splitting methods for American option pricing

Quadratic convergence for valuing American options using a penalty method

The Mathematics of financial derivatives
*Cambridge University Press*
Combining the discretization with the time derivatives

\[ \mathcal{L} V_{i,j} \bigg|_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{h_T} + O(h_T) \]

\[ V_{j+1} = V_j + h_T [\mathcal{L} V_j + O(h_T)] \]
\[ = V_j + h_T \left[ CV_j + D_j + O(h_T + h_S^2) \right] \]

\[ \mathcal{L} V_{i,j+1} = \frac{V_{i,j+1} - V_{i,j}}{h_T} + O(h_T) \]

\[ V_j = V_{j+1} - h_T \left[ \mathcal{L} V_{j+1} + O(h_T) \right] \]
\[ = V_{j+1} - h_T \left[ CV_{j+1} + D_{j+1} + O(h_T + h_S^2) \right] \]
\[ V_{j+1} = \left[ I - h_T C \right]^{-1} \left[ V_j + h_T (D_{j+1} + O(h_T + h_S^2)) \right] \]
The two methods can be used in combination, with weight $\theta \in [0, 1]$. Choosing $\theta = \frac{1}{2}$ gives the Crank-Nicolson Scheme. Once again, the equation can be rewritten in matrix form:

$$\frac{V_{i,j+1} - V_{i,j}}{h_\tau} = \theta \mathcal{L}V_{i,j+1} + (1 - \theta)\mathcal{L}V_{i,j} + O(h_\tau)$$

$$[I - h_\tau \theta C] V_{j+1} - h_\tau \theta D_{j+1} =$$

$$[I + h_\tau (1 - \theta) C] V_j + h_\tau (1 - \theta) D_j + h_\tau O(h_\tau + h_\tau^2).$$

Combining the terms into single matrices

$$C_{im} V_{j+1} + D_{im,j+1} = C_{ex} V_j + D_{ex,j} + h_\tau O(h_\tau + h_\tau^2)$$

where

$$C_{im} = I - h_\tau \theta C, \quad D_{im,j+1} = -h_\tau \theta D_{j+1}$$

$$C_{ex} = I + h_\tau (1 - \theta) C, \quad D_{ex,j} = h_\tau (1 - \theta) D_j.$$
We define a general (possibly non-uniform) grid using similar notation:

\[
\begin{align*}
0 &= S_0 < S_1 < \ldots < S_{N_S} = S_{\text{max}} \\
0 &= \tau_0 < \tau_1 < \ldots < \tau_{N_\tau} = T \\
h_{S_i} &= S_{i+1} - S_i, \quad h_{\tau_j} = \tau_{j+1} - \tau_j \\
V_{i,j} &= V(S_i, \tau_j) \\
\forall i &= 0, 1, \ldots, N_S, \ j = 0, 1, \ldots, N_\tau
\end{align*}
\] (34)
Choice of Grids

A non-uniform grid scheme can be defined by a monotonically increasing function

$$W : [0, S_{\text{max}}] \mapsto [0, S_{\text{max}}]$$ (35)

where $W$ maps a uniform grid $[0, h, 2h, \cdots, N_h h]$ to a non-uniform grid $[0, S_1, S_2, \cdots, S_{N_S}]$.

One possible grid arises from the mapping function

$$W_1(S) = \left[ 1 + \frac{\sinh \left(b \left( \frac{S}{S_{\text{max}}} - a \right) \right)}{\sinh(ba)} \right] K$$ (36)

where $a$ is a parameter determining the concentration near the strike $K$, and $b$ is chosen such that $W_1(S_{\text{max}}) = S_{\text{max}}$. Choosing larger $a$ generates more concentration near the strike.
Choice of Grids

Similar to the price derivative, we consider the method suggested in Forsyth and Vetzal (2002) where the time step is selected adaptively

\[
h_{\tau_{j+1}} \sim \left[ \frac{\partial V}{\partial \tau} \right]_{j}^{-1}
\]  

(37)

The step size is chosen based on the previous time step and the time derivative

\[
h_{\tau_{j+1}} = h_{\tau_{j}} \min_{i} \left[ d_{\text{norm}} \frac{\max(d_{0}, |V_{i,j+1}|, |V_{i,j}|)}{|V_{i,j+1} - V_{i,j}|} \right]
\]  

(38)

where \(d_{\text{norm}}\) is the target relative change per time step, and \(d_{0}\) is chosen as a scale so that \(h_{\tau}\) is not reduced due to the value \(V_{i,j}\) being close to zero.
Choice of Grids

The finite difference approximations in the price dimension then take on a different form:

$$\left. \frac{\partial V}{\partial S} \right|_{i,j} = \frac{h_{S_i}^2 V_{i+1,j} + (h_{S_{i+1}}^2 - h_{S_i}^2) V_{i,j} - h_{S_{i+1}}^2 V_{i-1,j}}{h_{S_i}(h_{S_i} + h_{S_{i+1}})h_{S_{i+1}}} + O(h_{S_i} \cdot h_{S_{i+1}})$$

(39a)

$$\left. \frac{\partial^2 V}{\partial S^2} \right|_{i,j} = \frac{2h_{S_i} V_{i+1,j} - 2(h_{S_{i+1}} + h_{S_i}) V_{i,j} + 2h_{S_{i+1}} V_{i-1,j}}{h_{S_i}(h_{S_i} + h_{S_{i+1}})h_{S_{i+1}}} + O(h_{S_{i+1}} - h_{S_i}) + O(\max\{h_{S_{i+1}}^2, h_{S_i}^2\}).$$

(39b)
### Results

#### Penalty Iteration on a non-uniform grid.

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<th>$N_{it}$</th>
<th>$V$</th>
<th>$V_{Change}$</th>
<th>$V_{Ratio}$</th>
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#### Operator Splitting on a non-uniform grid.

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*Table: Penalty Iteration on a non-uniform grid.*

*Table: Operator Splitting on a non-uniform grid.*
### Results

<table>
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<tr>
<th>$S_f$</th>
<th>$S_{f,\text{Ratio}}$</th>
<th>$\Delta$</th>
<th>$\Delta_{\text{Ratio}}$</th>
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**Table**: Penalty Iteration on a non-uniform grid.

<table>
<thead>
<tr>
<th>$S_f$</th>
<th>$S_{f,\text{Ratio}}$</th>
<th>$\Delta$</th>
<th>$\Delta_{\text{Ratio}}$</th>
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**Table**: Operator Splitting on a non-uniform grid.