Numerical PDE Methods for Pricing the American Put Option

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Overview

Financial Options

An **option** is a type of derivative contract that gives the holder the right, but not the obligation, to buy or sell an underlying security at a certain price (**strike price**) and time (**expiry time**) in the future. Specifically a **put** option allows the holder to sell.

An **European** option only allows exercise of the right at the expiry date. This form of options admit an analytical solution formula.

An **American** option allows exercise at any time before and including the expiry date. This type of options does not admit a formula, therefore we resort to numerical methods.

Assume the asset price S follows

$$dS = \mu S dt + \sigma S dW \tag{1}$$

where μ is the drift, σ is the volatility, and W is the standard Wiener process.

For simplification of later equations, a change of variable $\tau = T - t$ is introduced. It can be shown that, if S follows (1), the price $V(S, \tau)$ of an European option satisfies the **Black-Scholes** (BS) equation

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \equiv \mathcal{L}V \tag{2}$$

where r is the risk-free interest rate.

Initial and Boundary Conditions

The price of an option at expiry ($\tau = 0$) is determined by a **payoff function** $\Phi(S)$

$$V_{put}(S, \tau)|_{\tau=0} = \Phi_{put}(S) = \max(K - S, 0). \tag{3}$$

The S variable, considered as space variable, spans the domain $[0, \infty)$. The prices of European put options at the left and right boundary of the spatial domain are given by

$$V_{put}(0, \tau) = Ke^{-r\tau}, \quad V_{put}(S, \tau) \xrightarrow[S \to \infty]{} 0 \tag{4}$$

respectively.

American Options

For American options, there is an additional key constraint due to the ability to exercise at anytime before expiry T, which adds complexity to the problem:

$$V(S, \tau) \geq \Phi(S), \quad 0 \leq \tau \leq T. \tag{5}$$

This leads to a decision boundary at $S_f(\tau)$, where for each τ it is optimal to exercise on one side of the boundary, and optimal to hold on the other.

American Options

More specifically, the restriction can be reformulated as a linear complementarity problem (LCP)

$$\begin{aligned} \frac{\partial V}{\partial \tau} - \mathcal{L}V &\geq 0, \\ V - \Phi &\geq 0, \\ (\frac{\partial V}{\partial \tau} - \mathcal{L}V)(V - \Phi) &= 0. \end{aligned} \tag{6}$$

with new boundary conditions

$$V_{put}(0, \tau) = K, \quad V_{put}(S, \tau) \xrightarrow[S \to \infty]{} 0. \tag{7}$$

American Options

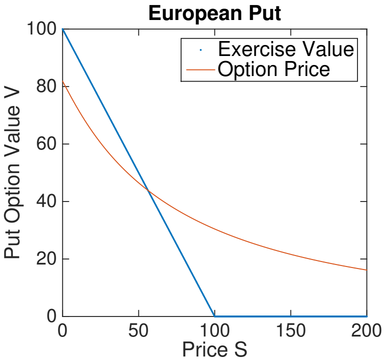


Figure: The price of an American put option follows different shapes on the two sides of the free boundary $S_f(\tau) = 51.57$.

Finite Difference Method

Instead of working with the full continuous price function $V(S, \tau)$, the function is sampled at specific points that form a grid,

$$V_{i,j} = V(S_i, \tau_j) \tag{8}$$

where a uniform grid is defined as

$$\begin{aligned} h_S &= \frac{S_{\max}}{N_S}, \quad S_i = ih_S, i = 0, 1, \ldots, N_S \\ h_\tau &= \frac{T}{N_\tau}, \quad \tau_j = jh_\tau, j = 0, 1, \ldots, N_\tau. \end{aligned} \tag{9}$$

The boundary condition to the right is approximated for large $S_{\max}$

$$V_{put}(S_{\max}, \tau_j) \approx 0. \tag{10}$$

We consider non-uniform grids later.

Finite Difference Method

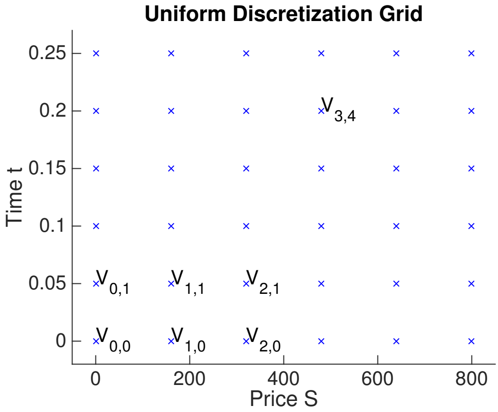


Figure: A sample uniform discretization grid with $S_{\max} = 800$, $T = 0.25$, $N_S = 5$, and $N_\tau = 5$.

Given the discretization grid points, the derivatives can be approximated by

$$\left.\frac{\partial V}{\partial \tau}\right|_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{h_\tau} + O(h_\tau) \tag{11a}$$

$$\left.\frac{\partial V}{\partial \tau}\right|_{i,j+1} = \frac{V_{i,j+1} - V_{i,j}}{h_\tau} + O(h_\tau) \tag{11b}$$

$$\left.\frac{\partial V}{\partial S}\right|_{i,j} = \frac{V_{i+1,j} - V_{i-1,j}}{2h_S} + O(h_S^2) \tag{11c}$$

$$\left.\frac{\partial^2 V}{\partial S^2}\right|_{i,j} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h_S^2} + O(h_S^2). \tag{11d}$$

In the above discretization schemes, $O(h_\tau)$ and $O(h_S^2)$ represent the residual terms.

Finite Difference Method

Similarly, the $\mathcal{L}V$ operator can be discretized as

$$
\begin{aligned}
\mathcal{L}V|_{i,j} &= (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV)\Big|_{i,j} \\
&= \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{2h_S} + rS_i \frac{V_{i+1,j} - V_{i-1,j}}{2h_S} - rV_{i,j} + O(h_S^2) \\
&= c_{i,i+1}V_{i+1,j} + c_{i,i}V_{i,j} + c_{i,i-1}V_{i-1,j} + O(h_S^2)
\end{aligned}
\tag{12}
$$

where $c_{i,k}$ are scalars

Finite Difference Method

The previous equation can be vectorized, and rewritten in matrix form as

$$\mathcal{L}\mathbf{V}_j = \mathbf{C}\mathbf{V}_j + \mathbf{D}_j + O(h_S^2) \tag{13}$$

with

$$\begin{aligned}
\mathbf{V}_j &= \begin{bmatrix} V_{1,j} & V_{2,j} & \cdots & V_{N_S-1,j} \end{bmatrix}^T \\
\mathbf{D}_j &= \begin{bmatrix} c_{1,0} V_{0,j} & 0 & \cdots & 0 & c_{N_S-1,N_S} V_{N_S,j} \end{bmatrix}^T \\
\mathbf{C} &= \begin{bmatrix} c_{1,1} & c_{1,2} & 0 & 0 & \cdots & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & c_{N_S-1,N_S-2} & c_{N_S-1,N_S-1} \end{bmatrix}
\end{aligned} \tag{14}$$

where the value of $\mathbf{D}_j$ is given by the boundary conditions, therefore known a priori.

Finite Difference Method

To summarize we converted the Black-Scholes PDE into a linear system of matrices

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\ \Rightarrow \quad \mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1} &= \mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j} + h_\tau O(h_\tau + h_S^2) \end{aligned} \tag{15}$$

where

$$\begin{aligned} \mathbf{C}_{im} &= \mathbf{I} - \frac{1}{2}h_\tau \mathbf{C}, \qquad & \mathbf{D}_{im,j+1} &= -\frac{1}{2}h_\tau \mathbf{D}_{j+1} \\ \mathbf{C}_{ex} &= \mathbf{I} + \frac{1}{2}h_\tau \mathbf{C}, & \mathbf{D}_{ex,j} &= \frac{1}{2}h_\tau \mathbf{D}_j. \end{aligned} \tag{16}$$

Finite Difference Method

The discretization of (6) also yields a matrix form for the LCP as

$$\begin{aligned}
(\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_{j} + \mathbf{D}_{ex,j}) &\geq 0, \\
\mathbf{V}_{j} - \mathbf{\Phi} &\geq 0, \\
\left[(\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_{j} + \mathbf{D}_{ex,j})\right] \cdot \left[\mathbf{V}_{j} - \mathbf{\Phi}\right] &= 0
\end{aligned} \tag{17}$$

Choice of Grids

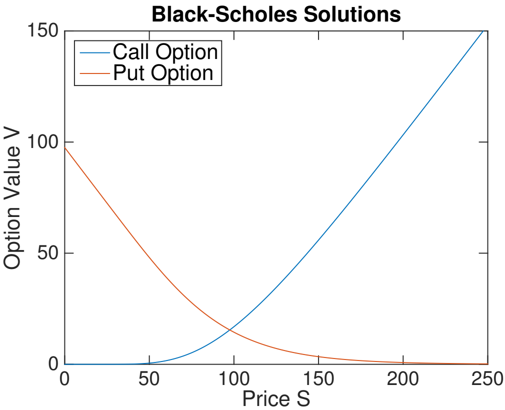


Figure: Typical solutions to the Black-Scholes PDE when strike price $K = 100$.

Choice of Grids

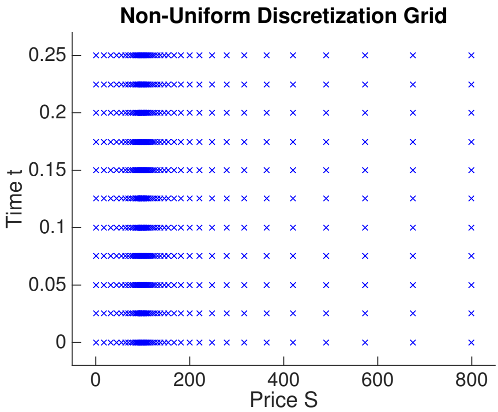


Figure: A sample non-uniform price discretization grid with $S_{\max} = 800$, $K = 100$, $N_S = 50$, and $a = 0.4$ under the $W_1(S)$ scheme combined with a uniform time discretization with $T = 0.25$ and $N_\tau = 10$.

Choice of Grids

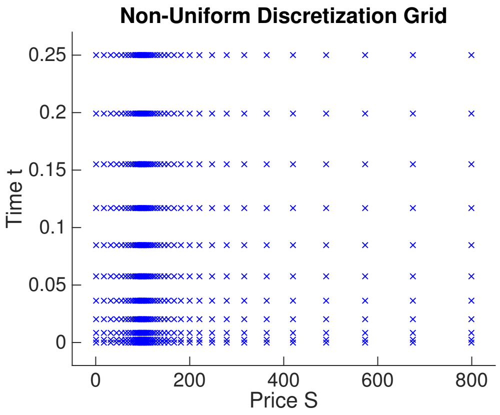
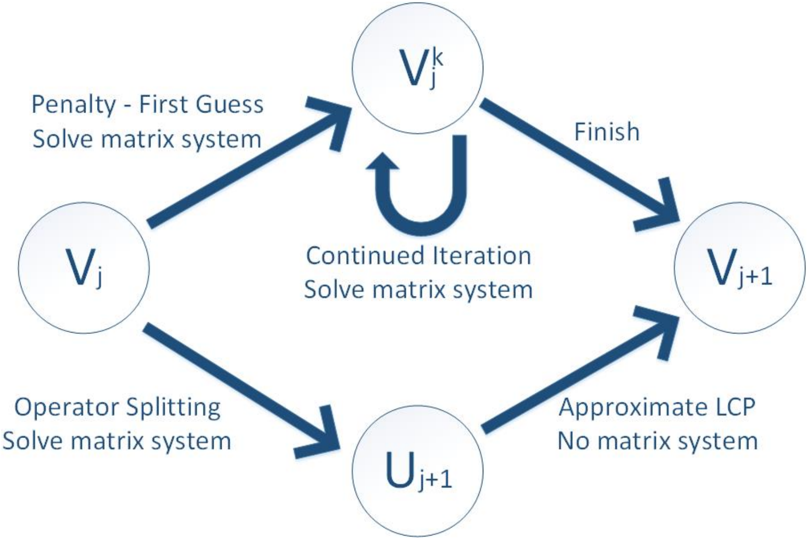


Figure: A sample non-uniform price discretization grid with $S_{\max} = 800$, $K = 100$, $N_S = 50$, and $a = 0.4$ under the $W_1(S)$ scheme combined with an adaptive non-uniform time discretization with $T = 0.25$, $N_\tau = 10$, and $d_{norm} = 0.2$

The discretization of (6) also yields a matrix form for the LCP as

$$\begin{aligned} (\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j}) &\geq 0, \\ \mathbf{V}_j - \mathbf{\Phi} &\geq 0, \\ \left[(\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j})\right] \cdot \left[\mathbf{V}_j - \mathbf{\Phi}\right] &= 0 \end{aligned} \tag{18}$$

Penalty Iteration vs. Operator Splitting



Discrete Penalty

The penalty method in Forsyth and Vetzal (2002) adds a term that punishes the difference arising from not satisfying the constraint. The formulation results in the relation

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V + \rho \max(\Phi - V, 0) \tag{19}$$

where ρ is the penalty parameter, and is typically chosen to be a large positive number. In the discrete case, a penalty term is placed on the next step at $\mathbf{V}_{j+1}$, resulting in the relation

$$\left[\mathbf{C}_{im} + \mathbf{P}_{j+1}\right]\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1} = \mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j} + \mathbf{P}_{j+1}\boldsymbol{\Phi} \tag{20}$$

where $\mathbf{P}_{j+1}$ is the penalty matrix, with the (i, n)th element defined as

$$\left[\mathbf{P}_j\right]_{i,n} = \begin{cases} \rho & \text{if } i = n \text{ and } V_{i,j} < \Phi \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Discrete Penalty

Let the iteration occur at the $(j+1)$th time step, define $\mathbf{V}^k$ as the kth estimate of $\mathbf{V}_{j+1}$, and $\mathbf{P}^k$ as the penalty for $\mathbf{V}^k$.

The iteration algorithm then follows:
Penalty Iteration for American Options

$$\begin{aligned} &\text{Initialize } \mathbf{V}^0 = \mathbf{V}_j \\ &\text{For } k = 1, 2, \cdots \\ &\quad \text{solve } \left[\mathbf{C}_{im} + \mathbf{P}^k\right]\mathbf{V}^k + \mathbf{D}_{im,j+1} = \mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j} + \mathbf{P}^k\mathbf{\Phi} \\ &\quad \text{if } \left[\max_i \frac{|\mathbf{V}^{k+1} - \mathbf{V}^k|}{\max(1, |\mathbf{V}^{k+1}|)} < \frac{1}{\rho}\right] \text{ or } \left[\mathbf{P}^{k+1} = \mathbf{P}^k\right] \text{ quit} \\ &\text{EndFor} \\ &\mathbf{V}_{j+1} = \mathbf{V}^k \end{aligned} \tag{22}$$

Operator Splitting

In contrast to the penalty iteration, the operator splitting method used in Ikonen and Toivanen (2004) is a direct solver. Similar to the penalty method, operator splitting also adds a term to resolve the inequality arising from the LCP ($\frac{\partial V}{\partial \tau} - \mathcal{L}V \geq 0$).

The auxiliary term is defined as

$$\lambda \equiv \frac{\partial V}{\partial \tau} - \mathcal{L}V \qquad (23)$$

Once again, the discretized LCP can be rewritten in matrix form

$$(\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j}) - h_\tau \boldsymbol{\lambda}_{j+1} = 0 \qquad (24)$$

where $\mathbf{V}_{j+1}$ and $\boldsymbol{\lambda}_{j+1}$ are both unknown vectors.

Operator Splitting

Instead of solving the previous equation, an intermediate term $\mathbf{U}_j$ is introduced such that,

$$(\mathbf{C}_{im}\mathbf{U}_{j+1} + \mathbf{D}_{im,j+1}) - (\mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j}) - h_\tau \boldsymbol{\lambda}_j = 0 \tag{25a}$$

$$\mathbf{C}_{im}(\mathbf{V}_{j+1} - \mathbf{U}_{j+1}) - h_\tau(\boldsymbol{\lambda}_{j+1} - \boldsymbol{\lambda}_j) = 0 \tag{25b}$$

Next (25b) is approximated by assuming the unknowns in $\mathbf{V}_{j+1}$ are independent, hence $\mathbf{C}_{im}$ becomes the identity matrix, and we get

$$\mathbf{V}_{j+1} - \mathbf{U}_{j+1} - h_\tau(\boldsymbol{\lambda}_{j+1} - \boldsymbol{\lambda}_j) = 0 \tag{26}$$

Operator Splitting

As a result of the simplification, the LCP constraints force each $\lambda_{i,j+1}$ to take on one of two values

$$\lambda_{i,j+1} = \begin{cases} 0 & \text{if } V_{i,j+1} > \Phi(S_i) \\ \lambda_{i,j} - \frac{1}{h_\tau}[U_{i,j+1} - \Phi(S_i)] & \text{if } V_{i,j+1} = \Phi(S_i) \end{cases} \quad (27)$$

Hence, we can find the intermediate term $\mathbf{U}_{j+1}$, the auxiliary term $\lambda_{i,j+1}$, and the price term $\mathbf{V}_{j+1}$ explicitly

$$\begin{aligned} \mathbf{U}_{j+1} &= \mathbf{C}_{im}^{-1}\left[\mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j} + \mathbf{D}_{im,j+1} + h_\tau \lambda_j\right] \\ \lambda_{i,j+1} &= \max\left(\lambda_{i,j} - \frac{1}{h_\tau}[U_{i,j+1} - \Phi(S_i)], 0\right) \quad (28) \\ \mathbf{V}_{j+1} &= \mathbf{U}_{j+1} + h_\tau(\boldsymbol{\lambda}_{j+1} - \boldsymbol{\lambda}_j) \end{aligned}$$

Results

Numerical Experiments were conducted with the following paramters on an American Put option:

$$\begin{aligned} K &= 100 \\ T &= 0.25 \\ \sigma &= 0.8 \\ r &= 0.1 \\ S_{\text{max}} &= 800 \\ \rho &= 1e6 \end{aligned} \tag{29}$$

Results - Choice of Methods

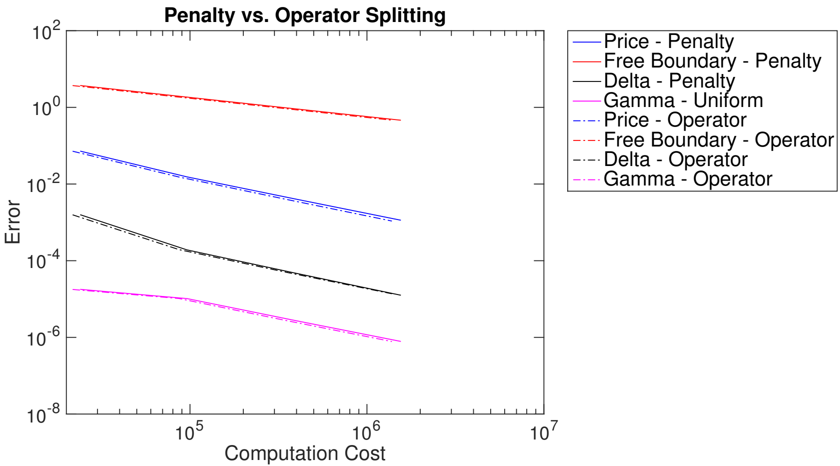


Figure: Convergence Property of the Penalty Iteration and Operator Splitting methods with Uniform grids.

Results - Choice of Grids

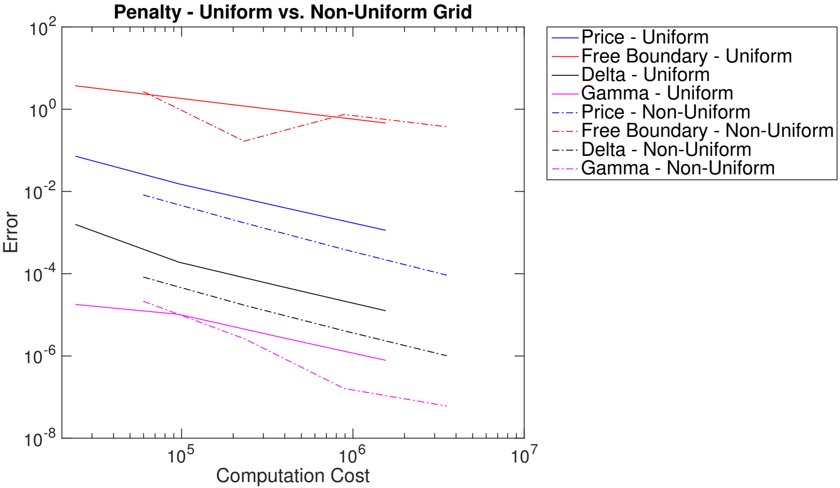


Figure: Convergence Property of the Penalty Iteration with Uniform and Non-uniform grids.

Future Work

Higher Dimensions

Since the penalty iteration requires more than one solution of a linear system per time step, its computational cost is expected to scale faster in multiple dimensions than the operator splitting method, which requires only one linear system solution per time step.

Optimal Grid

There is no available guideline to choose the optimal parameters for a non-uniform grid.

Other Derivative Products

Asian options, barrier options etc.

Questions or Comments?

S. Ikonen and J. Toivanen (2004)
Operator splitting methods for American option pricing
Appl. Math. Let. 17, 809 – 814.

P. A. Forsyth and K. R. Vetzal (2002)
Quadratic convergence for valuing American options using a penalty method
SIAM J. Sci. Comput. 23(6), 2095–2122

P. Wilmott and S. Howison and J. Dewynne (1995)
The Mathematics of financial derivatives
Cambridge University Press

Finite Difference Method

Combining the discretization with the time derivatives

$$
\begin{aligned}
\mathcal{L}V\big|_{i,j} &= \frac{V_{i,j+1} - V_{i,j}}{h_\tau} + O(h_\tau) \\
\mathbf{V}_{j+1} &= \mathbf{V}_j + h_\tau[\mathcal{L}\mathbf{V}_j + O(h_\tau)] \\
&= \mathbf{V}_j + h_\tau \left[\mathbf{C}\mathbf{V}_j + \mathbf{D}_j + O(h_\tau + h_S^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V\big|_{i,j+1} &= \frac{V_{i,j+1} - V_{i,j}}{h_\tau} + O(h_\tau) \\
\mathbf{V}_j &= \mathbf{V}_{j+1} - h_\tau\left[\mathcal{L}\mathbf{V}_{j+1} + O(h_\tau)\right] \\
&= \mathbf{V}_{j+1} - h_\tau \left[\mathbf{C}\mathbf{V}_{j+1} + \mathbf{D}_{j+1} + O(h_\tau + h_S^2)\right] \\
\mathbf{V}_{j+1} &= \left[\mathbf{I} - h_\tau\mathbf{C}\right]^{-1}\left[\mathbf{V}_j + h_\tau\left(\mathbf{D}_{j+1} + O(h_\tau + h_S^2)\right)\right]
\end{aligned}
\tag{30}
$$

Finite Difference Method

The two methods can be used in combination, with weight $\theta \in [0, 1]$. Choosing $\theta = \frac{1}{2}$ gives the Crank-Nicolson Scheme. Once again, the equation can be rewritten in matrix form:

$$\begin{aligned} \frac{V_{i.j+1} - V_{i,j}}{h_\tau} &= \theta \mathcal{L} V|_{i.j+1} + (1-\theta)\mathcal{L} V|_{i.j} + O(h_\tau) \\ [\mathbf{I} - h_\tau \theta \mathbf{C}]\, \mathbf{V}_{j+1} - h_\tau \theta \mathbf{D}_{j+1} &= \\ &[\mathbf{I} + h_\tau (1-\theta)\mathbf{C}]\mathbf{V}_j + h_\tau (1-\theta)\mathbf{D}_j + h_\tau O(h_\tau + h_S^2). \end{aligned} \tag{31}$$

Combining the terms into single matrices

$$\mathbf{C}_{im}\mathbf{V}_{j+1} + \mathbf{D}_{im,j+1} = \mathbf{C}_{ex}\mathbf{V}_j + \mathbf{D}_{ex,j} + h_\tau O(h_\tau + h_S^2) \tag{32}$$

where

$$\begin{aligned} \mathbf{C}_{im} &= \mathbf{I} - h_\tau \theta \mathbf{C}, \quad & \mathbf{D}_{im,j+1} &= -h_\tau \theta \mathbf{D}_{j+1} \\ \mathbf{C}_{ex} &= \mathbf{I} + h_\tau (1-\theta)\mathbf{C}, \quad & \mathbf{D}_{ex,j} &= h_\tau (1-\theta)\mathbf{D}_j. \end{aligned} \tag{33}$$

Choice of Grids

We define a general (possibly non-uniform) grid using similar notation:

$$
\begin{aligned}
0 &= S_0 < S_1 < \ldots < S_{N_S} = S_{\max} \\
0 &= \tau_0 < \tau_1 < \ldots < \tau_{N_\tau} = T \\
h_{S_i} &= S_{i+1} - S_i, \quad h_{\tau_j} = \tau_{j+1} - \tau_j \\
V_{i,j} &= V(S_i, \tau_j) \\
\forall i &= 0, 1, \ldots, N_S, \ j = 0, 1, \ldots, N_\tau
\end{aligned}
\tag{34}
$$

Choice of Grids

A non-uniform grid scheme can be defined by a monotonically increasing function

$$W : [0, S_{\max}] \mapsto [0, S_{\max}] \tag{35}$$

where W maps a uniform grid $[0, h_S, 2h_S, \cdots, N_S h_S]$ to a non-uniform grid $[0, S_1, S_2, \cdots, S_{N_S}]$.

One possible grid arises from the mapping function

$$W_1(S) = \left[1 + \frac{\sinh\left(b(\frac{S}{S_{\max}} - a)\right)}{\sinh(ba)}\right] K \tag{36}$$

where a is a parameter determining the concentration near the strike K, and b is chosen such that $W_1(S_{\max}) = S_{\max}$. Choosing larger a generates more concentration near the strike.

Choice of Grids

Similar to the price derivative, we consider the method suggested in Forsyth and Vetzal (2002) where the time step is selected adaptively

$$h_{\tau_{j+1}} \sim \left[\frac{\partial V}{\partial \tau} \right]_j^{-1} \tag{37}$$

The step size is chosen based on the previous time step and the time derivative

$$h_{\tau_{j+1}} = h_{\tau_j} \min_i \left[d_{norm} \frac{\max(d_0, |V_{i,j+1}|, |V_{i,j}|)}{|V_{i,j+1} - V_{i,j}|} \right] \tag{38}$$

where d_{norm} is the target relative change per time step, and d_0 is chosen as a scale so that h_τ is not reduced due to the value $V_{i,j}$ being close to zero.

Choice of Grids

The finite difference approximations in the price dimension then take on a different form:

$$\left.\frac{\partial V}{\partial S}\right|_{i,j} = \frac{h_{S_i}^2 V_{i+1,j} + (h_{S_{i+1}}^2 - h_{S_i}^2) V_{i,j} - h_{S_{i+1}}^2 V_{i-1,j}}{h_{S_i}(h_{S_i} + h_{S_{i+1}}) h_{S_{i+1}}} + O(h_{S_i} \cdot h_{S_{i+1}}) \tag{39a}$$

$$\left.\frac{\partial^2 V}{\partial S^2}\right|_{i,j} = \frac{2h_{S_i} V_{i+1,j} - 2(h_{S_{i+1}} + h_{S_i}) V_{i,j} + 2h_{S_{i+1}} V_{i-1,j}}{h_{S_i}(h_{S_i} + h_{S_{i+1}}) h_{S_{i+1}}} + O(h_{S_{i+1}} - h_{S_i}) + O(\max\{h_{S_{i+1}}^2, h_{S_i}^2\}). \tag{39b}$$

Results

N_S	N_τ	N_{it}	V	V_{Change}	V_{Ratio}
55	61	75	14.657284	0.000000	0.00
109	109	140	14.673142	0.015857	0.00
217	207	268	14.677518	0.004377	3.62
433	404	527	14.678531	0.001012	4.32
865	798	1032	14.678791	0.000260	3.90
1729	1588	2051	14.678856	0.000066	3.95

Table: Penalty Iteration on a non-uniform grid.

N_S	N_τ	N_{it}	V	V_{Change}	V_{Ratio}
55	75	75	14.400654	0.000000	0.00
109	144	144	14.673477	0.272823	0.00
217	278	278	14.677701	0.004224	64.59
433	547	547	14.678621	0.000920	4.59
865	1085	1085	14.678832	0.000211	4.35
1729	2162	2162	14.678875	0.000042	4.98

Table: Operator Splitting on a non-uniform grid.

Results

S_f	$S_{f,Ratio}$	Δ	Δ_{Ratio}	Γ	Γ_{Ratio}
60.1793	0.00	-0.405892	0.00	0.01000937	0.00
54.3226	0.00	-0.405699	0.00	0.01002033	0.00
53.6584	8.82	-0.405646	3.70	0.01002302	4.07
53.0823	1.15	-0.405633	3.95	0.01002367	4.15
52.3214	0.76	-0.405630	3.90	0.01002383	4.01
51.9340	1.96	-0.405629	3.92	0.01002387	4.02

Table: Penalty Iteration on a non-uniform grid.

S_f	$S_{f,Ratio}$	Δ	Δ_{Ratio}	Γ	Γ_{Ratio}
60.1793	0.00	-0.407603	0.00	0.01020692	0.00
54.3226	0.00	-0.405708	0.00	0.01002064	0.00
53.6584	8.82	-0.405651	33.17	0.01002314	-74.67
53.0823	1.15	-0.405635	3.64	0.01002372	4.29
52.3214	0.76	-0.405631	3.40	0.01002385	4.31
51.9340	1.96	-0.405629	3.17	0.01002388	4.88

Table: Operator Splitting on a non-uniform grid.